

# Solving a characteristic Cauchy problem

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## Abstract

In this paper we give a meaning to the nonlinear characteristic Cauchy problem for the Wave Equation in base form by replacing it by a family of non-characteristic problems in an appropriate algebra of generalized functions. We prove existence of a solution and we precise how it depends on the choice made. We also check that in the classical case (non-characteristic) our new solution coincides with the classical one.

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**Key words :** algebras of generalized functions; nonlinear partial differential equations; Characteristic Cauchy problem; Wave Equation.

## 1 Introduction

The goal of this paper is to study the solution to the following characteristic Cauchy problem

$$(P_C) \left\{ \begin{array}{l} \frac{\partial^2 u}{\partial x \partial y} = F(.,., u), \\ u|_{\gamma} = \varphi, \\ \frac{\partial u}{\partial y}|_{\gamma} = \psi, \end{array} \right.$$

where  $\gamma$  is a curve supposed to be a  $C^\infty$  manifold characteristic for the problem,  $\varphi$  and  $\psi$  being smooth functions defined in  $\gamma$ . The initial values have to be defined as restrictions of functions (even in a generalized sense) given in  $\gamma$  whose equation is  $y = f(x)$ .

As there is no classical solution here, we will look for a solution in a broader context, using the framework of generalized functions [1], [10], [18], [19]. They are an efficient tool to solve nonlinear problems as in [16], [17]. The general idea goes as follows. The characteristic problem is approached by a one-parameter family of classical smooth problems by deforming the characteristic

curve  $y = f(x)$  into a family of non-characteristic ones  $y = f_\varepsilon(x)$ ; we then get a one-parameter family of classical solutions. That is where the framework of generalized functions is used ; by means of this regularization, we define an associated generalized problem and we interpret this family of solutions as a generalized solution itself. Indeed a generalized function can be defined as a one-parameter family of smooth functions satisfying some asymptotical growth restrictions [19].

More precisely we will take  $(f_\varepsilon)_\varepsilon$  to be equivalent to  $f$  for some sense in an appropriate algebra of generalized functions. Furthermore, by imposing some restrictions on the asymptotical growth of the  $f_\varepsilon$ , we are able to prove that the generalized solution depends solely on the class of  $(f_\varepsilon)_\varepsilon$  as a generalized function, not on the particular representative. We also prove that in the non-characteristic smooth case, the generalized solution provided by our method coincides (in the sense of generalized functions) with the classical smooth solution.

The plan of this article is as follows. This section is followed by section 2 which briefly introduces the generalized algebras with our application in mind. We define a generalized differential problem associated to the ill posed classical one, then we proceed in section 3 with the proof of the existence of the generalized solution. In our case this amounts to prove that, provided a set of restrictions on the curve and the chosen deformation, the one-parameter family of solutions satisfy the required asymptotical growth. Subsection 3.3 is devoted to prove that the generalized solution does not depend on the representative of  $(\gamma_\varepsilon)_\varepsilon$ .

Then in section 4 we compute a few examples of characteristic equations, and make the link with distributional solutions.

## 2 Algebras of generalized functions

We recall briefly here the definition of the  $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -algebras upon which the remaining of this paper is based. We follow here the expositions found in [13], [14], [15], [16], [17] and [8]. We do not intend to properly define and explain  $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -algebras here, we rather want to fix the notations we will use in the latter sections. We refer the reader to the references.

The formalism described here, the  $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -algebras, is well suited for partial differential equations because of its parametric nature; this will become clear in the next section.

### 2.1 The presheaves of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -algebras

#### 2.1.1 Definitions

Take

- $\Lambda$  a set of indices;
- $A$  a solid subring of the ring  $\mathbb{K}^\Lambda$ , ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ), that is  $A$  has the following stability property: whenever  $(|s_\lambda|)_\lambda \leq (r_\lambda)_\lambda$  (i.e. for any  $\lambda$ ,  $|s_\lambda| \leq r_\lambda$ )

for any pair  $((s_\lambda)_\lambda, (r_\lambda)_\lambda) \in \mathbb{K}^\Lambda \times |A|$ , it follows that  $(s_\lambda)_\lambda \in A$ , with  $|A| = \{(|r_\lambda|)_\lambda : (r_\lambda)_\lambda \in A\}$ ;

- $I_A$  an solid ideal of  $A$  with the same property;
- $\mathcal{E}$  a sheaf of  $\mathbb{K}$ -topological algebras on a topological space  $X$ , such that for any open set  $\Omega$  in  $X$ , the algebra  $\mathcal{E}(\Omega)$  is endowed with a family  $\mathcal{P}(\Omega) = (p_i)_{i \in I(\Omega)}$  of seminorms satisfying

$$\forall i \in I(\Omega), \exists (j, k, C) \in I(\Omega) \times I(\Omega) \times \mathbb{R}_+^*, \forall f, g \in \mathcal{E}(\Omega) : p_i(fg) \leq Cp_j(f)p_k(g).$$

Assume that

- For any two open subsets  $\Omega_1, \Omega_2$  of  $X$  such that  $\Omega_1 \subset \Omega_2$ , we have  $I(\Omega_1) \subset I(\Omega_2)$  and if  $\rho_1^2$  is the restriction operator  $\mathcal{E}(\Omega_2) \rightarrow \mathcal{E}(\Omega_1)$ , then, for each  $p_i \in \mathcal{P}(\Omega_1)$ , the seminorm  $\tilde{p}_i = p_i \circ \rho_1^2$  extends  $p_i$  to  $\mathcal{P}(\Omega_2)$ ;
- For any family  $\mathcal{F} = (\Omega_h)_{h \in H}$  of open subsets of  $X$  if  $\Omega = \cup_{h \in H} \Omega_h$ , then, for each  $p_i \in \mathcal{P}(\Omega)$ ,  $i \in I(\Omega)$ , there exists a finite subfamily  $\Omega_1, \dots, \Omega_{n(i)}$  of  $\mathcal{F}$  and corresponding seminorms  $p_1 \in \mathcal{P}(\Omega_1), \dots, p_{n(i)} \in \mathcal{P}(\Omega_{n(i)})$ , such that, for each  $u \in \mathcal{E}(\Omega)$ ,

$$p_i(u) \leq p_1(u|_{\Omega_1}) + \dots + p_{n(i)}(u|_{\Omega_{n(i)}}).$$

Set

$$\begin{aligned} \mathcal{X}_{(A, \mathcal{E}, \mathcal{P})}(\Omega) &= \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda : \forall i \in I(\Omega), ((p_i(u_\lambda))_\lambda) \in |A|\}, \\ \mathcal{N}_{(I_A, \mathcal{E}, \mathcal{P})}(\Omega) &= \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda : \forall i \in I(\Omega), (p_i(u_\lambda))_\lambda \in |I_A|\}, \\ \mathcal{C} &= A/I_A. \end{aligned}$$

One can prove that  $\mathcal{X}_{(A, \mathcal{E}, \mathcal{P})}$  is a sheaf of subalgebras of the sheaf  $\mathcal{E}^\Lambda$  and  $\mathcal{N}_{(I_A, \mathcal{E}, \mathcal{P})}$  is a sheaf of ideals of  $\mathcal{X}_{(A, \mathcal{E}, \mathcal{P})}$  [14]. Moreover, the constant sheaf  $\mathcal{X}_{(A, \mathbb{K}, |\cdot|)}/\mathcal{N}_{(I_A, \mathbb{K}, |\cdot|)}$  is exactly the sheaf  $\mathcal{C} = A/I_A$ , and if  $\mathbb{K} = \mathbb{R}$ ,  $\mathcal{C}$  will be denoted  $\overline{\mathbb{R}}$ .

**Definition 1.** We call presheaf of  $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -algebra the factor presheaf of algebras over the ring  $\mathcal{C} = A/I_A$

$$\mathcal{A} = \mathcal{X}_{(A, \mathcal{E}, \mathcal{P})}/\mathcal{N}_{(I_A, \mathcal{E}, \mathcal{P})}.$$

We denote by  $[u_\lambda]$  the class in  $\mathcal{A}(\Omega)$  defined by the representative  $(u_\lambda)_{\lambda \in \Lambda} \in \mathcal{X}_{(A, \mathcal{E}, \mathcal{P})}(\Omega)$ .

### 2.1.2 Overgenerated rings

See [7]. Let  $B_p = \{(r_{n, \lambda})_\lambda \in (\mathbb{R}_+^*)^\Lambda : n = 1, \dots, p\}$  and  $B$  be the subset of  $(\mathbb{R}_+^*)^\Lambda$  obtained as rational functions with coefficients in  $\mathbb{R}_+^*$ , of elements in  $B_p$  as variables. Define

$$A = \{(a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \exists (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda\}.$$

**Definition 2.** In the above situation, we say that  $A$  is overgenerated by  $B_p$  (and it is easy to see that  $A$  is a solid subring of  $\mathbb{K}^\Lambda$ ). If  $I_A$  is some solid ideal of  $A$ , we also say that  $\mathcal{C} = A/I_A$  is overgenerated by  $B_p$ .

**Example 3.** For example, as a “canonical” ideal of  $A$ , we can take

$$I_A = \{(a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \forall (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda\}.$$

**Remark 4.** We can see that with this definition  $B$  is stable by inverse.

### 2.1.3 Relationship with distribution theory

Let  $\Omega$  an open subset of  $\mathbb{R}^n$ . The space of distributions  $\mathcal{D}'(\Omega)$  can be embedded into  $\mathcal{A}(\Omega)$ . If  $(\varphi_\lambda)_{\lambda \in (0,1]}$  is a family of mollifiers  $\varphi_\lambda(x) = \frac{1}{\lambda^n} \varphi(\frac{x}{\lambda})$ ,  $x \in \mathbb{R}^n$ ,  $\int \varphi(x) dx = 1$  and if  $T \in \mathcal{D}'(\mathbb{R}^n)$ , the convolution product family  $(T * \varphi_\lambda)_\lambda$  is a family of smooth functions slowly increasing in  $\frac{1}{\lambda}$ . So, for  $\Lambda = (0, 1]$ , we shall choose the subring  $A$  overgenerated by some  $B_p$  of  $(\mathbb{R}_+^*)^\Lambda$  containing the family  $(\lambda)_\lambda$ , [4], [20].

### 2.1.4 The association process

We assume that  $\Lambda$  is left-filtering for a given partial order relation  $\prec$ . We denote by  $\Omega$  an open subset of  $X$ ,  $E$  a given sheaf of topological  $\mathbb{K}$ -vector spaces containing  $\mathcal{E}$  as a subsheaf,  $a$  a given map from  $\Lambda$  to  $\mathbb{K}$  such that  $(a(\lambda))_\lambda = (a_\lambda)_\lambda$  is an element of  $A$ . We also assume that

$$\mathcal{N}_{(I_A, \mathcal{E}, \mathcal{P})}(\Omega) \subset \left\{ (u_\lambda)_\lambda \in \mathcal{X}_{(A, \mathcal{E}, \mathcal{P})}(\Omega) : \lim_{E(\Omega), \Lambda} u_\lambda = 0 \right\}.$$

**Definition 5.** We say that  $u = [u_\lambda]$  and  $v = [v_\lambda] \in \mathcal{E}(\Omega)$  are  $a$ - $E$  associated if

$$\lim_{E(\Omega), \Lambda} a_\lambda (u_\lambda - v_\lambda) = 0.$$

That is to say, for each neighborhood  $V$  of 0 for the  $E$ -topology, there exists  $\lambda_0 \in \Lambda$  such that  $\lambda \prec \lambda_0 \implies a_\lambda (u_\lambda - v_\lambda) \in V$ . We write

$$u \underset{E(\Omega)}{\overset{a}{\sim}} v.$$

**Remark 6.** We can also define an association process between  $u = [u_\lambda]$  and  $T \in \mathcal{E}(\Omega)$  by writing simply

$$u \sim T \iff \lim_{E(\Omega), \Lambda} u_\lambda = T.$$

Taking  $E = \mathcal{D}'$ ,  $\mathcal{E} = C^\infty$ ,  $\Lambda = (0, 1]$ , we recover the association process defined in the literature ([2], [3]).

## 2.2 Algebraic framework for our problem

Set  $\mathcal{E} = C^\infty$ ,  $X = \mathbb{R}^d$  for  $d = 1, 2$ ,  $E = \mathcal{D}'$  and  $\Lambda$  a set of indices,  $\lambda \in \Lambda$ . For any open set  $\Omega$ , in  $\mathbb{R}^d$ ,  $\mathcal{E}(\Omega)$  is endowed with the  $\mathcal{P}(\Omega)$  topology of uniform convergence of all derivatives on compact subsets of  $\Omega$ . This topology may be defined by the family of the seminorms

$$P_{K,l}(u_\lambda) = \sup_{|\alpha| \leq l} P_{K,\alpha}(u_\lambda) \quad \text{with} \quad P_{K,\alpha}(u_\lambda) = \sup_{x \in K} |D^\alpha u_\lambda(x)|, \quad K \Subset \Omega$$

where the notation  $K \Subset \mathbb{R}^2$  means that  $K$  is a compact subset of  $\mathbb{R}^2$  and  $D^\alpha = \frac{\partial^{\alpha_1 + \dots + \alpha_d}}{\partial z_1^{\alpha_1} \dots \partial z_d^{\alpha_d}}$  for  $z = (z_1, \dots, z_d) \in \Omega$ ,  $l \in \mathbb{N}$ ,  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ . Let  $A$  be a subring of the ring  $\mathbb{R}^\Lambda$  of family of reals with the usual laws. We consider a solid ideal  $I_A$  of  $A$ . Then we have

$$\begin{aligned} \mathcal{X}(\Omega) &= \{(u_\lambda)_\lambda \in [C^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N}, (P_{K,l}(u_\lambda))_\lambda \in |A|\}, \\ \mathcal{N}(\Omega) &= \{(u_\lambda)_\lambda \in [C^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N}, (P_{K,l}(u_\lambda))_\lambda \in |I_A|\}, \\ \mathcal{A}(\Omega) &= \mathcal{X}(\Omega)/\mathcal{N}(\Omega). \end{aligned}$$

The generalized derivation  $D^\alpha : u(= [u_\varepsilon]) \mapsto D^\alpha u = [D^\alpha u_\varepsilon]$  provides  $\mathcal{A}(\Omega)$  with a differential algebraic structure (cf [6]).

**Example 7.** Set  $\Lambda = (0, 1]$ . Consider

$$A = \mathbb{R}_M^\Lambda = \{(m_\lambda)_\lambda \in \mathbb{R}^\Lambda : \exists p \in \mathbb{R}_+^*, \exists C \in \mathbb{R}_+^*, \exists \mu \in (0, 1], \forall \lambda \in (0, \mu], |m_\lambda| \leq C\lambda^{-p}\}$$

and the ideal

$$I_A = \{(m_\lambda)_\lambda \in \mathbb{R}^\Lambda : \forall q \in \mathbb{R}_+^*, \exists D \in \mathbb{R}_+^*, \exists \mu \in (0, 1], \forall \lambda \in (0, \mu], |m_\lambda| \leq D\lambda^q\}.$$

In this case we denote  $\mathcal{X}^s(\Omega) = \mathcal{X}(\Omega)$  and  $\mathcal{N}^s(\Omega) = \mathcal{N}(\Omega)$ . The sheaf of factor algebras  $\mathcal{G}^s(\cdot) = \mathcal{X}^s(\cdot)/\mathcal{N}^s(\cdot)$  is called the sheaf of simplified Colombeau algebras.  $\mathcal{G}^s(\mathbb{R}^d)$  is the simplified Colombeau algebra of generalized functions [2], [3].

We have the analogue of theorem 1.2.3. of [10] for  $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -algebras. We suppose here that  $\Lambda$  is left filtering and give this proposition for  $\mathcal{A}(\mathbb{R}^2)$ , although it is valid in more general situations.

**Proposition 8.** Let  $B$  be the set introduced in Definition 2 and assume that there exists  $(a_\lambda)_\lambda \in B$  with  $\lim_{\lambda \rightarrow 0} a_\lambda = 0$ . Consider  $(u_\lambda)_\lambda \in \mathcal{X}(\mathbb{R}^2)$  such that

$$\forall K \Subset \mathbb{R}^2, (P_{K,0}(u_\lambda))_\lambda \in |I_A|.$$

Then  $(u_\lambda)_\lambda \in \mathcal{N}(\mathbb{R}^2)$ .

We refer the reader to [7] and [5] for a detailed proof.

**Definition 9.** *Tempered generalized functions, [10], [22], [21]. For  $f \in C^\infty(\mathbb{R}^n)$ ,  $r \in \mathbb{Z}$  and  $m \in \mathbb{N}$ , we put*

$$\mu_{r,m}(f) = \sup_{x \in \mathbb{R}^n, |\alpha| \leq m} (1 + |x|)^r |\mathcal{D}^\alpha f(x)|.$$

*The space of functions with slow growth is*

$$\mathcal{O}_M(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \forall m \in \mathbb{N}, \exists q \in \mathbb{N}, \mu_{-q,m}(f) < +\infty\}.$$

**Definition 10.** *We put*

$$\begin{aligned} \mathcal{X}_\tau(\mathbb{R}^n) &= \{(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^n)^{(0,1]} : \forall m \in \mathbb{N}, \exists q \in \mathbb{N}, \exists N \in \mathbb{N}, \mu_{-q,m}(f_\varepsilon) = O(\varepsilon^{-N}) \text{ } (\varepsilon \rightarrow 0)\}, \\ \mathcal{N}_\tau(\mathbb{R}^n) &= \{(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^n)^{(0,1]} : \forall m \in \mathbb{N}, \exists q \in \mathbb{N}, \forall p \in \mathbb{N}, \mu_{-q,m}(f_\varepsilon) = O(\varepsilon^p) \text{ } (\varepsilon \rightarrow 0)\}. \end{aligned}$$

$\mathcal{X}_\tau(\mathbb{R}^n)$  is a subalgebra of  $\mathcal{O}_M(\mathbb{R}^n)^{(0,1]}$  and  $\mathcal{N}_\tau(\mathbb{R}^n)$  an ideal of  $\mathcal{X}_\tau(\mathbb{R}^n)$ . The algebra  $\mathcal{G}_\tau(\mathbb{R}^n) = \mathcal{X}_\tau(\mathbb{R}^n) / \mathcal{N}_\tau(\mathbb{R}^n)$  is called the algebra of tempered generalized functions. The generalized derivation  $\mathcal{D}^\alpha : u = [u_\varepsilon] \mapsto \mathcal{D}^\alpha u = [\mathcal{D}^\alpha u_\varepsilon]$  provides  $\mathcal{G}_\tau(\mathbb{R}^n)$  with a differential algebraic structure.

If  $u$  is a generalized function of the variable  $x \in \mathbb{R}^2$  and  $F \in C^\infty(\mathbb{R}^3, \mathbb{R})$ , we extend the notation  $F(\cdot, \cdot, u)$  in the following way:

**Definition 11.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^2$  and  $F \in C^\infty(\Omega \times \mathbb{R}, \mathbb{R})$ . We say that the algebra  $\mathcal{A}(\Omega)$  is stable under  $F$  if the following two conditions are satisfied:*

- *For each  $K \Subset \mathbb{R}^2$ ,  $l \in \mathbb{N}$  and  $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$ , there is a positive finite sequence  $C_0, \dots, C_l$ , such that*

$$P_{K,l}(F(\cdot, \cdot, u_\varepsilon)) \leq \sum_{i=0}^l C_i (P_{K,l}(u_\varepsilon))^i.$$

- *For each  $K \Subset \mathbb{R}^2$ ,  $l \in \mathbb{N}$ ,  $(v_\varepsilon)_\varepsilon$  and  $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$ , there is a positive finite sequence  $D_1, \dots, D_l$  such that*

$$P_{K,l}(F(\cdot, \cdot, v_\varepsilon) - F(\cdot, \cdot, u_\varepsilon)) \leq \sum_{j=1}^l D_j (P_{K,l}(v_\varepsilon - u_\varepsilon))^j.$$

**Remark 12.** *If  $\mathcal{A}(\Omega)$  is stable under  $F$  then, for all  $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$  and  $(i_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$ , we have  $(F(\cdot, \cdot, u_\varepsilon))_\varepsilon \in \mathcal{X}(\Omega)$ ;  $(F(\cdot, \cdot, u_\varepsilon + i_\varepsilon) - F(\cdot, \cdot, u_\varepsilon))_\varepsilon \in \mathcal{N}(\Omega)$ .*

### 2.2.1 Generalized operator associated to a stability property

If  $\mathcal{A}(\mathbb{R}^2)$  is stable under  $F$ , for  $u = [u_\varepsilon] \in \mathcal{A}(\mathbb{R}^2)$ ,  $[F(\cdot, \cdot, u_\varepsilon)]$  is a well defined element of  $\mathcal{A}(\mathbb{R}^2)$  (i.e. not depending on the representative  $(u_\varepsilon)_\varepsilon$  of  $u$ ). This leads to the following:

**Definition 13.** *If  $\mathcal{A}(\mathbb{R}^2)$  is stable under  $F$ , the operator*

$$\mathcal{F} : \mathcal{A}(\mathbb{R}^2) \rightarrow \mathcal{A}(\mathbb{R}^2), \quad u = [u_\varepsilon] \mapsto [F(\cdot, \cdot, u_\varepsilon)]$$

*is called the generalized operator associated to  $F$ . See [7].*

### 2.2.2 Generalized restriction mappings

Set  $(f_\varepsilon)_\varepsilon$  be a family of functions in  $C^\infty(\mathbb{R})$ . For each  $g \in C^\infty(\mathbb{R}^2)$  set

$$R_\varepsilon(g) : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}), \quad f_\varepsilon \mapsto (x \mapsto g(x, f_\varepsilon(x))).$$

The family  $(R_\varepsilon)_\varepsilon$  map  $(C^\infty(\mathbb{R}^2))^\Lambda$  into  $(C^\infty(\mathbb{R}))^\Lambda$ .

**Definition 14.** *The family of smooth function  $(f_\varepsilon)_\varepsilon$  is compatible with second side restriction if*

$$\begin{aligned} \forall (u_\varepsilon)_\varepsilon &\in \mathcal{X}(\mathbb{R}^2), \quad (u_\varepsilon(\cdot, f_\varepsilon(\cdot)))_\varepsilon \in \mathcal{X}(\mathbb{R}); \\ \forall (i_\varepsilon)_\varepsilon &\in \mathcal{N}(\mathbb{R}^2), \quad (i_\varepsilon(\cdot, f_\varepsilon(\cdot)))_\varepsilon \in \mathcal{N}(\mathbb{R}). \end{aligned}$$

Clearly, if  $u = [u_\varepsilon] \in \mathcal{A}(\mathbb{R}^2)$  then  $[u_\varepsilon(\cdot, f_\varepsilon(\cdot))]$  is a well defined element of  $\mathcal{A}(\mathbb{R})$  (i.e. not depending on the representative of  $u$ .) This leads to the following:

**Definition 15.** *If the family of smooth function  $(f_\varepsilon)_\varepsilon$  is compatible with second side restriction, the mapping*

$$\mathcal{R} : \mathcal{A}(\mathbb{R}^2) \rightarrow \mathcal{A}(\mathbb{R}), \quad u = [u_\varepsilon] \mapsto [u_\varepsilon(\cdot, f_\varepsilon(\cdot))] = [R_\varepsilon(u_\varepsilon)]$$

*is called the generalized second side restriction mapping associated to the family  $(f_\varepsilon)_\varepsilon$ .*

**Remark 16.** *The previous process generalizes the standard one defining the restriction of the generalized function  $u = [u_\varepsilon] \in \mathcal{A}(\mathbb{R}^2)$  to the manifold  $\{y = f(x)\}$  obtained when taking  $f_\varepsilon = f$  for each  $\varepsilon \in \Lambda$ .*

First let us state a useful definition used throughout this article:

**Definition 17.** [10] *Let  $(f_\varepsilon)_\varepsilon$  be a family of  $C^\infty(\mathbb{R}^n)$  functions. This family is  $c$ -bounded if for all compact set  $K \subset \mathbb{R}^n$  it exists another compact set  $L \subset \mathbb{R}^n$  such that  $f_\varepsilon(K) \subset L$  for all  $\varepsilon$  ( $L$  is independent of  $\varepsilon$ ).*

**Proposition 18.** *Assume that:*

- (i) *For each  $K \Subset \mathbb{R}$ , it exists  $K' \Subset \mathbb{R}$  such that, for all  $\varepsilon \in \Lambda$ ,  $f_\varepsilon(K) \subset K'$ ,*
- (ii)  *$(f_\varepsilon)_\varepsilon$  belongs to  $\mathcal{X}(\mathbb{R})$ .*

*Then the family  $(f_\varepsilon)_\varepsilon$  is compatible with restriction.*

*Proof.* Take  $(u_\varepsilon)_\varepsilon$  (resp.  $(i_\varepsilon)_\varepsilon$ ) in  $\mathcal{X}(\mathbb{R}^2)$  (resp.  $\mathcal{N}(\mathbb{R}^2)$ ) and set  $v_\varepsilon(x) = u_\varepsilon(x, f_\varepsilon(x))$ . From (i) we have

$$\begin{aligned} p_{K,0}(v_\varepsilon) &\leq p_{K \times K',0}(u_\varepsilon), \\ p_{K,1}(v_\varepsilon) &\leq p_{K \times K',(1,0)}(u_\varepsilon) + p_{K \times K',(0,1)}(u_\varepsilon) p_{K,1}(f_\varepsilon). \end{aligned}$$

By induction we can see that for each  $K \Subset \mathbb{R}$ , and each  $l \in \mathbb{N}$ ,  $p_{K,l}(v_\varepsilon)$  is estimated by sums or products of terms like  $p_{K \times K',(n,m)}(u_\varepsilon)$  for  $n + m \leq l$ , or  $p_{K,k}(f_\varepsilon)$  for  $k \leq l$ . Then, from (ii),  $p_{K,l}(v_\varepsilon)$  is in  $|A|$ . Similarly, setting  $j_\varepsilon(x) = i_\varepsilon(x, f_\varepsilon(x))$  leads to  $p_{K,l}(j_\varepsilon) \in |I_A|$ . Then  $(u_\varepsilon(\cdot, f_\varepsilon(\cdot)))_\varepsilon$  (resp.  $(i_\varepsilon(\cdot, f_\varepsilon(\cdot)))_\varepsilon$ ) belongs to  $\mathcal{X}(\mathbb{R})$  (resp.  $\mathcal{N}(\mathbb{R})$ ).  $\square$

### 3 Existence of solutions for a characteristic Cauchy problem in $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -algebras

We will use the notations found in [8] and [9].

#### 3.1 A generalized differential problem associated to the ill posed classical one

Our goal is to give a meaning to the characteristic Cauchy problem formally written as

$$(P_{form}) \left\{ \begin{array}{l} \frac{\partial^2 u}{\partial x \partial y} = F(\cdot, \cdot, u), \\ u|_{\gamma} = \varphi, \\ \frac{\partial u}{\partial y} \Big|_{\gamma} = \psi, \end{array} \right.$$

where the data  $\psi, \varphi$  are smooth functions given on a characteristic  $C^\infty$  manifold  $\gamma$  supposed to be a curve whose equation is  $y = f(x)$ .  $F$  is smooth in its arguments.

We don't have a classical surrounding in which we can pose (and a fortiori solve) the problem. In the sequel, by means of regularizing processes we will define an associated problem to  $(P_{form})$ .

$$(P_{gen}) \left\{ \begin{array}{l} \frac{\partial^2 u}{\partial x \partial y} = \mathcal{F}(u), \\ \mathcal{R}(u) = \varphi, \\ \mathcal{R}\left(\frac{\partial u}{\partial y}\right) = \psi \end{array} \right.$$

where  $u$  is searched in some convenient algebra  $\mathcal{A}(\mathbb{R}^2)$ ,  $\mathcal{F}, \mathcal{R}$  are defined as previously.

The idea is then to approach this Cauchy problem by a family of non-characteristic ones by replacing the characteristic curve  $\gamma$  by a family of smooth non-characteristic curves  $\gamma_\varepsilon$  whose equation is  $y = f_\varepsilon(x)$ . Moreover  $\gamma_\varepsilon$  is diffeomorphic to  $y = 0$ , which is a consequence of the following assumption

$$(H) : \left\{ \begin{array}{l} F \in C^\infty(\mathbb{R}^3, \mathbb{R}), \\ \forall K \Subset \mathbb{R}^2, \sup_{(x,y) \in K; z \in \mathbb{R}} |\partial_z F(x, y, z)| < \infty, \\ f_\varepsilon \text{ is defined and strictly increasing on } \mathbb{R} \text{ with image } \mathbb{R}, \\ \forall x \in \mathbb{R}, f'_\varepsilon(x) \neq 0. \end{array} \right.$$

In terms of representatives, and thanks to the stability and restriction hy-



pothesis, solving  $(P_{gen})$  amounts to find a family  $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^2)$  such that

$$\begin{cases} \frac{\partial^2 u_\varepsilon}{\partial x \partial y}(x, y) - F_\varepsilon(x, y, u_\varepsilon(x, y)) = i_\varepsilon(x, y), \\ u_\varepsilon(x, f_\varepsilon(x)) - \varphi(x) = j_\varepsilon(x), \\ \frac{\partial u_\varepsilon}{\partial y}(x, f_\varepsilon(x)) - \psi(x) = l_\varepsilon(x), \end{cases}$$

where  $(i_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ ,  $(j_\varepsilon)_\varepsilon, (l_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$ . Suppose we can find  $u_\varepsilon \in C^\infty(\mathbb{R}^2)$  verifying

$$\begin{cases} \frac{\partial^2 u_\varepsilon}{\partial x \partial y}(x, y) = F_\varepsilon(x, y, u_\varepsilon(x, y)), \\ u_\varepsilon(x, f_\varepsilon(x)) = \varphi(x), \\ \frac{\partial u_\varepsilon}{\partial y}(x, f_\varepsilon(x)) = \psi(x), \end{cases}$$

then, if we can prove that  $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^2)$ ,  $u = [u_\varepsilon]$  is a solution of  $(P_{gen})$ . If  $v = [v_\varepsilon]$  is another solution of  $(P_{gen})$  obtain by replacing  $\gamma$  by another family of smooth non-characteristic curves  $\gamma'_\varepsilon$  whose equation is  $y = g_\varepsilon(x)$ , this implies

$$\begin{cases} \frac{\partial^2 (v_\varepsilon - u_\varepsilon)}{\partial x \partial y}(x, y) - (F_\varepsilon(x, y, v_\varepsilon(x, y)) - F_\varepsilon(x, y, u_\varepsilon(x, y))) = a_\varepsilon(x, y), \\ v_\varepsilon(x, g_\varepsilon(x)) - u_\varepsilon(x, f_\varepsilon(x)) = b_\varepsilon(x), \\ \frac{\partial v_\varepsilon}{\partial y}(x, g_\varepsilon(x)) - \frac{\partial u_\varepsilon}{\partial y}(x, f_\varepsilon(x)) = c_\varepsilon(x), \end{cases}$$

where  $(a_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$  and  $(b_\varepsilon)_\varepsilon, (c_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$ . We have to prove that  $(v_\varepsilon - u_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$  if we intend to prove that the solution of  $(P_{gen})$  in the algebra  $\mathcal{A}(\mathbb{R}^2)$  does not depend on the representative of the class  $[f_\varepsilon]$ .

### 3.2 Notations, assumptions and results

Using the results of [9], with assumption  $(H)$  we have a unique smooth solution of  $(P_\varepsilon)$  which satisfies the following integral equation

$$u_\varepsilon(x, y) = u_{0,\varepsilon}(x, y) - \iint_{D(x, y, f_\varepsilon)} F(\xi, \eta, u_\varepsilon(\xi, \eta)) d\xi d\eta,$$

with  $u_{0,\varepsilon}(x, y) = \varphi(x) - \chi_\varepsilon(f_\varepsilon(x)) + \chi_\varepsilon(y)$ , where  $\chi_\varepsilon$  is a primitive of  $\psi \circ f_\varepsilon^{-1}$  and

$$D(x, y, f_\varepsilon) = \begin{cases} \{(\xi, \eta) / x \leq \xi \leq f_\varepsilon^{-1}(y), f_\varepsilon(\xi) \leq \eta \leq y\}, & \text{if } y \geq f_\varepsilon(x), \\ \{(\xi, \eta) / f_\varepsilon^{-1}(y) \leq \xi \leq x, y \leq \eta \leq f_\varepsilon(\xi)\}, & \text{if } y \leq f_\varepsilon(x). \end{cases}$$

#### Remarks, notations and hypothesis.

Each compact  $K \Subset \mathbb{R}^2$  is contained in some product  $[-a, a] \times [-b, b]$ . We define

$$\begin{cases} \beta_{K,\varepsilon} = \max(a, f_\varepsilon^{-1}(b)) \text{ and } \alpha_{K,\varepsilon} = \min(-a, f_\varepsilon^{-1}(-b)), \\ a_{K,\varepsilon} = 2 \max(\beta_{K,\varepsilon}, |\alpha_{K,\varepsilon}|), \\ K_\varepsilon = K_{1\varepsilon} \times K_2 \text{ with } K_{1\varepsilon} = [-a_{K,\varepsilon}/2, a_{K,\varepsilon}/2] \text{ and } K_2 = [-b, b] = [-c/2, c/2]. \end{cases} \quad (1)$$

By construction we have  $K \subset K_\varepsilon$ .

We also make the following assumptions to generate a convenient  $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -algebra adapted to our problem:

$$(H_1) \begin{cases} \forall (\varepsilon, \eta) \in (0, 1]^2, \forall K \in \mathbb{R}^2, \forall l \in \mathbb{N}, \exists \mu_{K,l} > 0, \exists M_\varepsilon > 0, \\ \sup_{(t,x,z) \in K_\varepsilon \times \mathbb{R}, |\alpha| \leq l} |D^\alpha F(t, x, z)| = M_{K,\varepsilon,l} \leq \mu_{K,l} M_\varepsilon. \\ \forall \varepsilon \in (0, 1], \forall K \in \mathbb{R}^2, \exists \nu_K > 0, \exists a_\varepsilon > 0, a_{K,\varepsilon} \leq \nu_K a_\varepsilon. \end{cases} \quad (2)$$

Particularly, we set

$$m_{K,\varepsilon} = \sup_{(t,x) \in K_\varepsilon; z \in \mathbb{R}} \left| \frac{\partial}{\partial z} F(t, x, z) \right| \leq \mu_{K,1} M_\varepsilon.$$

$$(H_2) \begin{cases} \exists (r_\varepsilon)_\varepsilon \in \mathbb{R}_*^{(0,1]} \text{ such that } \forall K_2 \in \mathbb{R}, \forall \alpha_2 \in \mathbb{N}, \exists D_2 = D_{K_2, \alpha_2, \varepsilon} \in \mathbb{R}_+^*, \exists p \in \mathbb{N}, \\ \max \left[ \sup_{K_2} |D^{\alpha_2} \varphi(f_\varepsilon^{-1}(y))|, \sup_{K_2} |D^{\alpha_2} \chi_\varepsilon(y)| \right] \leq \frac{D_2}{(r_\varepsilon)^p}. \end{cases}$$

$$(H_3) \begin{cases} \mathcal{C} = A/I_A \text{ is overgenerated by the following elements of } \mathbb{R}_*^{(0,1]} \\ (\varepsilon)_\varepsilon, (r_\varepsilon)_\varepsilon, (M_\varepsilon)_\varepsilon, (\exp M_\varepsilon a_\varepsilon). \end{cases}$$

$$(H_4) \begin{cases} \mathcal{A}(\mathbb{R}^2) = \mathcal{X}(\mathbb{R}^2)/\mathcal{N}(\mathbb{R}^2) \text{ is built on } \mathcal{C} \text{ with} \\ (\mathcal{E}, \mathcal{P}) = (C^\infty(\mathbb{R}^2), (P_{K,l})_{K \in \mathbb{R}^2, l \in \mathbb{N}}) \\ \text{and } \mathcal{A}(\mathbb{R}^2) \text{ is stable under } F \text{ relatively to } \mathcal{C}. \end{cases}$$

$$(H_5) \quad f_\varepsilon \in C^\infty(\mathbb{R}), f_\varepsilon \text{ strictly increasing, } f_\varepsilon(\mathbb{R}) = \mathbb{R} \text{ and } \phi, \psi \in O_M(\mathbb{R})$$

$$(H_6) \quad (f_\varepsilon)_\varepsilon, (f_\varepsilon^{-1})_\varepsilon \in \mathcal{X}_\tau(\mathbb{R}), (f_\varepsilon)_\varepsilon \text{ is c-bounded and } \lim_{\varepsilon \xrightarrow{\mathcal{D}'(\mathbb{R})} 0} f_\varepsilon = f.$$

**Lemma 19.** *We have the following relations:*

$$(|\alpha_{K,\varepsilon}|)_\varepsilon, (|\beta_{K,\varepsilon}|)_\varepsilon, (|a_{K,\varepsilon}|)_\varepsilon \in |A|, \quad (3)$$

$$\forall \varepsilon, \forall (x, y) \in K_\varepsilon, D(x, y, f_\varepsilon) \subset K_\varepsilon. \quad (4)$$

First  $(f_\varepsilon^{-1})_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$  so  $(|\alpha_{K,\varepsilon}|)_\varepsilon, (|\beta_{K,\varepsilon}|)_\varepsilon \in |A|$  and then obviously  $(|a_{K,\varepsilon}|)_\varepsilon \in |A|$ . Next as  $(f_\varepsilon)_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$  we can find  $p \in \mathbb{N}$  such that  $\forall x, \varepsilon, |f_\varepsilon(x)| \leq \varepsilon^{-p}(1 + |x|)^p$  so we have

$$|\mu_\varepsilon| = |f_\varepsilon(\alpha_{K,\varepsilon})| \leq \varepsilon^{-p}(1 + |\alpha_{K,\varepsilon}|)^p \in |A|.$$

**Theorem 20.** *With the notations and the hypothesis of the above paragraph, if  $u_\varepsilon$  is the solution to the problem  $(P_\varepsilon)$  then problem  $(P_{gen})$  admits  $u = [u_\varepsilon]_{\mathcal{A}(\mathbb{R}^2)}$  as solution.*

**Proof.** We have:  $u_\varepsilon(x, y) = u_{0,\varepsilon}(x, y) - u_{1,\varepsilon}(x, y)$ , where

$$u_{0,\varepsilon}(x, y) = \varphi(x) - \chi_\varepsilon(f_\varepsilon(x)) + \chi_\varepsilon(y),$$

and

$$u_{1,\varepsilon}(x, y) = \iint_{D(x, y, f_\varepsilon)} F(\xi, \eta, u_\varepsilon(\xi, \eta)) d\xi d\eta.$$

We will actually prove that  $(P_{K_\varepsilon, n}(u_\varepsilon))_\varepsilon \in |A|$ .

First we have  $\chi'_\varepsilon = \psi \circ f_\varepsilon^{-1}$ ; as for  $f_\varepsilon^{-1}(K_2] = K_{1\varepsilon}$  and as  $\psi \in \mathcal{O}_M(\mathbb{R})$ ,  $(|\alpha_{K, \varepsilon}|)_\varepsilon, (|\beta_{K, \varepsilon}|)_\varepsilon \in |A|$  then

$$\forall l \in \mathbb{N}, (P_{K_2, l}(\chi_\varepsilon))_\varepsilon \in |A|.$$

Moreover as  $\varphi \in \mathcal{O}_M(\mathbb{R})$  we also have that

$$\forall l \in \mathbb{N}, (P_{K_{1\varepsilon}, l}(\varphi))_\varepsilon \in |A|$$

and as  $(\chi_\varepsilon \circ f_\varepsilon)' = f'_\varepsilon \psi$  and  $(f'_\varepsilon)_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$  we can conclude that

$$\forall l \in \mathbb{N}, (P_{K_\varepsilon, l}(u_{0, \varepsilon}))_\varepsilon \in |A|.$$

We have the following equality

$$F(x, y, u_\varepsilon(x, y)) - F(x, y, 0) = u_\varepsilon(x, y) \int_0^1 \frac{\partial F}{\partial z}(x, y, \theta u_\varepsilon(x, y)) d\theta$$

so that

$$|u_\varepsilon(x, y)| \leq \|u_{0, \varepsilon}\|_{K_\varepsilon} + \iint_{D(x, y, f_\varepsilon)} \left| u_\varepsilon(\xi, \eta) \int_0^1 \frac{\partial F}{\partial z}(x, y, \theta u_\varepsilon(x, y)) d\theta + F(x, y, 0) \right| d\xi d\eta.$$

Now if we define  $A(K_\varepsilon)$  denotes the area of  $K_\varepsilon$

$$C_\varepsilon = A(K_\varepsilon) \sup_{(x, y) \in D_\varepsilon} |F(x, y, 0)| + \|u_{0, \varepsilon}\|_{K_\varepsilon} \in |A|,$$

we get (remembering relation (4))

$$\forall (x, y) \in K_\varepsilon, |u_\varepsilon(x, y)| \leq C_\varepsilon + \iint_{D(x, y, f_\varepsilon)} \mu_{K, 1} M_\varepsilon |u_\varepsilon(\xi, \eta)| d\xi d\eta.$$

We define  $e_\varepsilon(y) = \sup_{x \in K_{1\varepsilon}} |u_\varepsilon(x, y)|$  and then for all  $y \in K_2$  we have

$$e_\varepsilon(y) \leq C_\varepsilon + \int_{-b}^b \mu_{K, 1} M_\varepsilon e_\varepsilon(\eta) d\eta,$$

so

$$e_\varepsilon(y) \leq C_\varepsilon \exp(2b\mu_{K, 1} M_\varepsilon) \text{ using Gronwall's lemma}$$

and then as  $C_\varepsilon \exp(2b\mu_{K, 1} M_\varepsilon) \in |A|$  we have proved that

$$(P_{K_\varepsilon, 0}(u_\varepsilon))_\varepsilon = (\|u_\varepsilon\|_{K_\varepsilon})_\varepsilon \in |A|.$$

Let us assume now that we have  $(P_{K_\varepsilon, n}(u_\varepsilon))_\varepsilon \in |A|$ , for all  $n \in \mathbb{N}$ .  
By successive derivations, for  $n \geq 1$  we obtain

$$\frac{\partial^{n+1} u_{1, \varepsilon}}{\partial x^{n+1}}(x, y) = - \sum_{j=0}^{n-1} C_n^j f_\varepsilon^{(n-j)}(x) \frac{\partial^j}{\partial x^j} F(x, f_\varepsilon(x), \varphi(x)) \quad (5)$$

$$+ \int_{f_\varepsilon(x)}^y \frac{\partial^n}{\partial x^n} F(x, \eta, u_\varepsilon(x, \eta)) d\eta. \quad (6)$$

First as  $(\alpha_\varepsilon)_\varepsilon, (\beta_\varepsilon)_\varepsilon \in |A|$  and  $\varphi \in \mathcal{O}_M(\mathbb{R})$  we have that  $(P_{K_{1\varepsilon}, j}(\varphi))_\varepsilon \in |A|$  because for any  $k \in \mathbb{N}$ , we can find  $p \in \mathbb{N}$  such that  $|\varphi^{(k)}(\alpha_\varepsilon)| \leq (1 + |\alpha_\varepsilon|)^p$ .  
Moreover  $f_\varepsilon \in \mathcal{X}_\tau$  then for all  $k$ , we can find  $p \in \mathbb{N}$  such that

$$\forall \varepsilon, \sup_{\mathbb{R}} (1 + |x|)^{-p} |f_\varepsilon^{(k)}(x)| \leq \varepsilon^{-p},$$

but then we have

$$\left\| f_\varepsilon^{(k)} \right\|_{K_{1\varepsilon}} \leq \max \{ (1 + |\alpha_\varepsilon|)^p, (1 + |\beta_\varepsilon|)^p \} \varepsilon^{-p} \in |A|$$

and as  $(P_{K_\varepsilon \times \mathbb{R}, j}(F))_\varepsilon \in |A|$  this takes care of the first term in (5). For the second term we compute first for  $n = 1$

$$\begin{aligned} & \int_{f_\varepsilon(x)}^y \frac{\partial}{\partial x} F(x, \eta, u_\varepsilon(x, \eta)) d\eta \\ &= \int_{f_\varepsilon(x)}^y \left( \frac{\partial F}{\partial x}(x, \eta, u_\varepsilon(x, \eta)) + \frac{\partial F}{\partial z}(x, \eta, u_\varepsilon(x, \eta)) \frac{\partial u_\varepsilon}{\partial x}(x, \eta) \right) d\eta, \end{aligned}$$

then we have

$$\left\| \int_{f_\varepsilon(x)}^y \frac{\partial}{\partial x} F(x, \eta, u_\varepsilon(x, \eta)) d\eta \right\|_{D_\varepsilon} \leq 2b P_{K_\varepsilon \times \mathbb{R}, 1}(F) (1 + P_{K_\varepsilon, 1}(u_\varepsilon)),$$

and then this is in  $|A|$  because of the hypothesis for  $F$ . Now for  $n = 2$  we have similarly

$$\left\| \int_{f_\varepsilon(x)}^y \frac{\partial^2}{\partial x^2} F(x, \eta, u_\varepsilon(x, \eta)) d\eta \right\|_{D_\varepsilon} \leq 2b P_{K_\varepsilon \times \mathbb{R}, 2}(F) (P_{K_\varepsilon, 1}(u_\varepsilon)^2 + P_{K_\varepsilon, 2}(u_\varepsilon) + 1)$$

and this is also in  $|A|$  as the induction hypothesis insures that  $(P_{K_\varepsilon, 2}(u_\varepsilon))_\varepsilon \in |A|$ .  
For  $n > 2$  the same calculation leads to more terms involving higher derivatives but each of them can be dealt with using the same kind of arguments.  
Similarly we have

$$\begin{aligned} \frac{\partial^{n+1} u_{1, \varepsilon}}{\partial y^{n+1}}(x, y) &= - \sum_{j=0}^{n-1} C_n^j (f_\varepsilon^{-1})^{(n-j)}(y) \frac{\partial^j}{\partial y^j} F(f_\varepsilon^{-1}(y), y, \varphi(f_\varepsilon^{-1}(y))) \\ &\quad - \int_x^{f_\varepsilon^{-1}(y)} \frac{\partial^n}{\partial y^n} F(\xi, y, u_\varepsilon(\xi, y)) d\xi. \end{aligned}$$

As  $f_\varepsilon^{-1} \in \mathcal{X}_\tau(\mathbb{R})$  and using the same argument as previously the first terms have their  $P_K$  norm in  $|A|$ .

Moreover we have

$$\left\| \int_x^{f_\varepsilon^{-1}(y)} \frac{\partial^n}{\partial y^n} F(\xi, y, u_\varepsilon(\xi, y)) d\xi \right\|_{K_\varepsilon} \leq a_{K, \varepsilon} \left\| \frac{\partial^n}{\partial y^n} F(x, y, u_\varepsilon(x, y)) \right\|_{K_\varepsilon}.$$

The same arguments apply here so that this term also has its  $P_{K_\varepsilon}$  norm in  $|A|$ . The proof for the other partial derivatives can be done along the same lines, thus finally we can conclude that  $(P_{K_\varepsilon, n+1}(u_\varepsilon))_\varepsilon \in |A|$  which concludes the induction.

It is to be observed that we prove more than we need for the existence alone, but we will definitely need this stronger statement when proving that this solution only depends on the class  $[f_\varepsilon]$  as we will have to compare different generalized solutions.

### 3.3 Generalized solutions only depend on the class $[f_\varepsilon]$

**Theorem 21.** *Under the same hypotheses as theorem 20, the generalized function  $u$  represented by the family  $(u_\varepsilon)_\varepsilon$  of solutions to Problems  $(P_\varepsilon)$ , does not depend on the choice of the representative  $(f_\varepsilon)_\varepsilon$  of the class  $f = [f_\varepsilon] \in \mathcal{G}_\tau(\mathbb{R})$ .*

**Remark 22.** *We need to consider tempered generalized functions for  $(f_\varepsilon)_\varepsilon$  as this counter-example shows:*

*Take  $f_\varepsilon(x) = \varepsilon x$  and  $g_\varepsilon = f_\varepsilon + n_\varepsilon$  where we have defined  $n_\varepsilon$  as an increasing  $C^\infty$  function satisfying*

$$n_\varepsilon(x) = \begin{cases} -1 & \text{if } x < -\frac{2}{\varepsilon}, \\ 0 & \text{if } x \in [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}], \\ 1 & \text{if } x > \frac{2}{\varepsilon}. \end{cases}$$

*Note that  $(n_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$  as for any compact  $K$ , it exists  $\varepsilon \in (0, 1)$  such that  $n_\varepsilon|_K \equiv 0$ . But it can easily be checked that  $n_\varepsilon \notin \mathcal{N}_\tau(\mathbb{R})$ . As  $f_\varepsilon$  is strictly increasing then  $g_\varepsilon = f_\varepsilon + n_\varepsilon$  is also. We have  $f_\varepsilon^{-1}(y) = y/\varepsilon$  so  $(f_\varepsilon^{-1})_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$ , moreover it is easy to choose  $n_\varepsilon$  so that  $(g_\varepsilon^{-1})_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$ . Now we have*

$$g_\varepsilon^{-1}(y) = \begin{cases} (y-1)/\varepsilon & \text{if } y > 3 \\ y/\varepsilon & \text{if } y \in [-1, 1]. \end{cases}$$

*So that  $\|f_\varepsilon^{-1} - g_\varepsilon^{-1}\|_{[3,4]} = 1/\varepsilon$  which proves that  $(f_\varepsilon^{-1} - g_\varepsilon^{-1})_\varepsilon \notin \mathcal{N}(\mathbb{R})$ .*

*Moreover if we now turn back to the wave equation and we set  $\psi(x) = x$  and  $F = 0$ , the 2 solutions  $(u_\varepsilon)_\varepsilon$  and  $(v_\varepsilon)_\varepsilon$  corresponding to  $(f_\varepsilon)_\varepsilon$  and  $(g_\varepsilon)_\varepsilon$  respectively are given by*

$$\begin{cases} u_\varepsilon(x, y) = \varphi(x) + \chi_\varepsilon^f(y) - \chi_\varepsilon^f(f_\varepsilon(x)), \\ v_\varepsilon(x, y) = \varphi(x) + \chi_\varepsilon^g(y) - \chi_\varepsilon^g(g_\varepsilon(x)), \end{cases}$$

so that we have

$$\frac{\partial(u_\varepsilon - v_\varepsilon)}{\partial y} = \psi(f_\varepsilon^{-1}(y)) - \psi(g_\varepsilon^{-1}(y)) = f_\varepsilon^{-1}(y) - g_\varepsilon^{-1}(y).$$

So  $\left(\frac{\partial(u_\varepsilon - v_\varepsilon)}{\partial y}\right)_\varepsilon \notin \mathcal{N}(\mathbb{R})$ . This proves that the usual equivalence is too coarse.

Before proving the theorem we need the following

**Lemma 23.** *Let  $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$  such that for every  $\varepsilon$ ,  $f_\varepsilon, g_\varepsilon$  are bijective and  $(f_\varepsilon^{-1})_\varepsilon, (g_\varepsilon^{-1})_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$ . If moreover  $(g_\varepsilon - f_\varepsilon)_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$  we have that*

$$(f_\varepsilon^{-1} - g_\varepsilon^{-1})_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$$

The proof will use the pointvalues characterization; so let us first define the following map (cf [10], 1.2)

$$\begin{aligned} \Theta : \mathcal{G}_\tau(\mathbb{R}) &\rightarrow \mathcal{F}(\overline{\mathbb{R}}) \\ [f_\varepsilon] &\mapsto f : \tilde{x} = [x_\varepsilon] \mapsto f(\tilde{x}) = [f_\varepsilon(x_\varepsilon)] \end{aligned}$$

where  $\overline{\mathbb{R}}$  denotes the field of generalized real numbers and  $\mathcal{F}(\overline{\mathbb{R}})$  is the set of map from  $\overline{\mathbb{R}}$  to  $\overline{\mathbb{R}}$ .

*Proof.* First  $\mathcal{G}_\tau(\mathbb{R})$  and  $\mathcal{F}(\overline{\mathbb{R}})$  can be endowed with a structure of unitary rings where the operations are addition and composition of functions (the unit is then the identity function). Let us prove that  $\Theta$  is a morphism between these rings; let  $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$  and  $f = \Theta([f_\varepsilon]), g = \Theta([g_\varepsilon]), h = \Theta([f_\varepsilon \circ g_\varepsilon])$  (note that  $(f_\varepsilon \circ g_\varepsilon)_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$ ):

$$\begin{aligned} \forall \tilde{x} = [x_\varepsilon] \in \overline{\mathbb{R}}, h(\tilde{x}) &= [(f_\varepsilon \circ g_\varepsilon)(x_\varepsilon)] = [f_\varepsilon(g_\varepsilon(x_\varepsilon))] \\ &= f(g(\tilde{x})) \\ &= \Theta([f_\varepsilon]) \circ \Theta([g_\varepsilon])(\tilde{x}) \end{aligned}$$

If we assume moreover that  $f_\varepsilon, g_\varepsilon$  are bijective,  $(f_\varepsilon^{-1})_\varepsilon, (g_\varepsilon^{-1})_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$  and  $(g_\varepsilon - f_\varepsilon)_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$ , we have that:

$$\begin{aligned} \text{Id} &= [f_\varepsilon \circ f_\varepsilon^{-1}] = \Theta([f_\varepsilon]) \circ \Theta([f_\varepsilon^{-1}]) \\ \text{Id} &= [f_\varepsilon^{-1} \circ f_\varepsilon] = \Theta([f_\varepsilon^{-1}]) \circ \Theta([f_\varepsilon]) \end{aligned}$$

So that  $\Theta([f_\varepsilon^{-1}]) = \Theta([f_\varepsilon])^{-1}$ . Now as  $[g_\varepsilon] = [f_\varepsilon]$ , we have that  $f = g$  so that  $f^{-1} = g^{-1}$  and then  $[f_\varepsilon^{-1}] = [g_\varepsilon^{-1}]$  which concludes the lemma.  $\square$

Now we can prove theorem 21.

We have

$$u_\varepsilon(x, y) = u_{0,\varepsilon}(x, y) - \iint_{D(x, y, f_\varepsilon)} F(\xi, \eta, u_\varepsilon(\xi, \eta)) d\xi d\eta,$$

where  $u_{0,\varepsilon}(x, y) = \chi_\varepsilon^f(y) - \chi_\varepsilon^f(f_\varepsilon(x)) + \varphi(x)$  and  $\chi_\varepsilon^f$  is a primitive of  $\psi \circ f_\varepsilon^{-1}$ . So we take  $(g_\varepsilon)_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$ , such that  $(f_\varepsilon - g_\varepsilon)_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$ ; let  $v = [v_\varepsilon]$  the corresponding generalized solution. Let us prove that  $u = v$ .

We will in fact have prove a slightly stronger statement for reasons that will be clear in the course of the proof

$$\forall K \in \mathbb{R}^2, \forall \alpha \in \mathbb{N}^2, (P_{K_\varepsilon, \alpha}(u_\varepsilon - v_\varepsilon))_\varepsilon \in I_A$$

Let us now fix  $K \in \mathbb{R}^2$ . We have

$$v_\varepsilon(x, y) = v_{0,\varepsilon}(x, y) - \iint_{D(x, y, g_\varepsilon)} F(\xi, \eta, v_\varepsilon(\xi, \eta)) d\xi d\eta,$$

where  $v_{0,\varepsilon}(x, y) = \chi_\varepsilon^g(y) - \chi_\varepsilon^g(g_\varepsilon(x)) + \varphi(x)$  and  $\chi_\varepsilon^g$  is a primitive of  $\psi \circ g_\varepsilon^{-1}$ . So we get

$$u_{0,\varepsilon}(x, y) - v_{0,\varepsilon}(x, y) = \chi_\varepsilon^f(y) - \chi_\varepsilon^g(y) - \chi_\varepsilon^f(f_\varepsilon(x)) + \chi_\varepsilon^g(g_\varepsilon(x)).$$

We compute

$$\frac{\partial}{\partial x} (-\chi_\varepsilon^f(f_\varepsilon(x)) + \chi_\varepsilon^g(g_\varepsilon(x))) = \psi(x) (f'_\varepsilon(x) - g'_\varepsilon(x)) \in \mathcal{N}_\tau(\mathbb{R}),$$

but this implies that  $\left\| \frac{\partial}{\partial x} (-\chi_\varepsilon^f(f_\varepsilon(x)) + \chi_\varepsilon^g(g_\varepsilon(x))) \right\|_{K_\varepsilon} \in I_A$ . Indeed we can find  $p \in \mathbb{N}$  such that for any  $m \in \mathbb{N}$  we have

$$\forall x \in \mathbb{R}, \frac{\partial}{\partial x} (-\chi_\varepsilon^f(f_\varepsilon(x)) + \chi_\varepsilon^g(g_\varepsilon(x))) \leq \varepsilon^m (1 + |x|)^p,$$

so

$$\left\| \frac{\partial}{\partial x} (-\chi_\varepsilon^f(f_\varepsilon(x)) + \chi_\varepsilon^g(g_\varepsilon(x))) \right\|_{K_\varepsilon} \leq \varepsilon^m \max \{ (1 + |a_{K,\varepsilon}/2|)^p \},$$

but  $((1 + |a_{K,\varepsilon}/2|)^p)_\varepsilon \in |A|$  thus we have obtained that

$$\left\| \frac{\partial}{\partial x} (-\chi_\varepsilon^f(f_\varepsilon(x)) + \chi_\varepsilon^g(g_\varepsilon(x))) \right\|_{K_\varepsilon} \in I_A.$$

Then we obtain that  $P_{K_\varepsilon, 1}(-\chi_\varepsilon^f(f_\varepsilon(x)) + \chi_\varepsilon^g(g_\varepsilon(x))) \in I_A$ . The proof is similar for higher derivatives.

Now as  $(f_\varepsilon^{-1} - g_\varepsilon^{-1})_\varepsilon \in \mathcal{N}(\mathbb{R})$  and  $\psi \in \mathcal{O}_M$  then  $(\psi \circ f_\varepsilon^{-1} - \psi \circ g_\varepsilon^{-1})_\varepsilon \in \mathcal{N}(\mathbb{R})$  and then we finally obtain that

$$\forall \alpha, (P_{K_\varepsilon, \alpha}(u_{0,\varepsilon} - v_{0,\varepsilon}))_\varepsilon \in I_A.$$

We compute

$$\begin{aligned}
& u_{1,\varepsilon}(x, y) - v_{1,\varepsilon}(x, y) \\
&= \int_{f_\varepsilon^{-1}(y)}^x \int_y^{f_\varepsilon(\xi)} F(\xi, \eta, u_\varepsilon(\xi, \eta)) d\eta d\xi - \int_{g_\varepsilon^{-1}(y)}^x \int_y^{g_\varepsilon(\xi)} F(\xi, \eta, v_\varepsilon(\xi, \eta)) d\eta d\xi \\
&= \int_{f_\varepsilon^{-1}(y)}^x \int_y^{f_\varepsilon(\xi)} [F(\xi, \eta, u_\varepsilon(\xi, \eta)) - F(\xi, \eta, v_\varepsilon(\xi, \eta))] d\eta d\xi \\
&\quad - \int_{g_\varepsilon^{-1}(y)}^{f_\varepsilon^{-1}(y)} \int_y^{g_\varepsilon(\xi)} F(\xi, \eta, v_\varepsilon(\xi, \eta)) d\eta d\xi - \int_{f_\varepsilon^{-1}(y)}^x \int_{f_\varepsilon(\xi)}^{g_\varepsilon(\xi)} F(\xi, \eta, v_\varepsilon(\xi, \eta)) d\eta d\xi.
\end{aligned}$$

As  $f_\varepsilon \circ g_\varepsilon^{-1} \equiv id \pmod{\mathcal{N}^s(\mathbb{R})}$ , we have

$$\begin{aligned}
\sup_{y \in [-b, b]} \left| \int_{f_\varepsilon^{-1}(y)}^{g_\varepsilon^{-1}(y)} \int_y^{g_\varepsilon(\xi)} F(\xi, \eta, v_\varepsilon(\xi, \eta)) d\eta d\xi \right| &\leq 2b \int_{f_\varepsilon^{-1}(y)}^{g_\varepsilon^{-1}(y)} \sup_{\eta \in [-b, b]} |F(\xi, \eta, v_\varepsilon(\xi, \eta))| d\xi \\
&\leq 2b \|f_\varepsilon^{-1} - g_\varepsilon^{-1}\|_{[-b, b]} \|F\|_{[\lambda_\varepsilon, \mu_\varepsilon] \times [-b, b] \times \mathbb{R}}
\end{aligned}$$

where

$$\begin{cases} \lambda_\varepsilon = \min\{f_\varepsilon^{-1}(-b), g_\varepsilon^{-1}(-b)\} \\ \mu_\varepsilon = \max\{f_\varepsilon^{-1}(b), g_\varepsilon^{-1}(b)\}, \end{cases}$$

which is negligible as  $f_\varepsilon^{-1} \sim g_\varepsilon^{-1}$  and  $\|F\|_{[\lambda_\varepsilon, \mu_\varepsilon] \times [-b, b] \times \mathbb{R}} \in |A|$ .

For the first derivative we have

$$\begin{aligned}
\frac{d}{dy} \left( \int_{f_\varepsilon^{-1}(y)}^{g_\varepsilon^{-1}(y)} \int_y^{g_\varepsilon(\xi)} F(\xi, \eta, v_\varepsilon(\xi, \eta)) d\eta d\xi \right) &= - \int_y^{g_\varepsilon(f_\varepsilon^{-1}(y))} F(f_\varepsilon^{-1}(y), \eta, v_\varepsilon(f_\varepsilon^{-1}(y), \eta)) d\eta \\
&\quad + \int_{f_\varepsilon^{-1}(y)}^{g_\varepsilon^{-1}(y)} F(\xi, y, v_\varepsilon(\xi, y)) d\xi.
\end{aligned}$$

And the same kind of arguments take care of those 2 terms. Now for the higher derivatives

$$\frac{d^2}{dy^2} \left( \int_{f_\varepsilon^{-1}(y)}^{g_\varepsilon^{-1}(y)} \int_y^{g_\varepsilon(\xi)} F(\xi, \eta, v_\varepsilon(\xi, \eta)) d\eta d\xi \right) \quad (7)$$

$$= -F(f_\varepsilon^{-1}(y), g_\varepsilon(f_\varepsilon^{-1}(y)), v_\varepsilon(f_\varepsilon^{-1}(y), g_\varepsilon(f_\varepsilon^{-1}(y)))) + F(f_\varepsilon^{-1}(y), y, v_\varepsilon(f_\varepsilon^{-1}(y), y)) \quad (8)$$

$$- \int_y^{g_\varepsilon(f_\varepsilon^{-1}(y))} \frac{d}{dy} (F(f_\varepsilon^{-1}(y), \eta, v_\varepsilon(f_\varepsilon^{-1}(y), \eta))) d\eta \quad (9)$$

$$+ F(g_\varepsilon^{-1}(y), y, v_\varepsilon(g_\varepsilon^{-1}(y), y)) - F(f_\varepsilon^{-1}(y), y, v_\varepsilon(f_\varepsilon^{-1}(y), y)) \quad (10)$$

$$+ \int_{f_\varepsilon^{-1}(y)}^{g_\varepsilon^{-1}(y)} \frac{d}{dy} (F(\xi, y, v_\varepsilon(\xi, y))) d\xi. \quad (11)$$



The hypotheses on  $F$  and the fact that  $(g_\varepsilon^{-1} - f_\varepsilon^{-1})_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$  takes care of the terms of lines (8) and (10). Let us now turn our attention to line (9). We have

$$\begin{aligned} & \int_y^{g_\varepsilon(f_\varepsilon^{-1}(y))} \frac{d}{dy} (F(f_\varepsilon^{-1}(y), \eta, v_\varepsilon(f_\varepsilon^{-1}(y), \eta))) d\eta \\ &= \int_y^{g_\varepsilon(f_\varepsilon^{-1}(y))} [(f_\varepsilon^{-1})'(y) \frac{\partial F}{\partial \xi}(f_\varepsilon^{-1}(y), \eta, v_\varepsilon(f_\varepsilon^{-1}(y), \eta)) \\ & \quad + (f_\varepsilon^{-1})'(y) \frac{\partial v_\varepsilon}{\partial x}(f_\varepsilon^{-1}(y), \eta) \frac{\partial F}{\partial z}(f_\varepsilon^{-1}(y), \eta, v_\varepsilon(f_\varepsilon^{-1}(y), \eta))] d\eta. \end{aligned}$$

As  $(g_\varepsilon \circ f_\varepsilon^{-1} - id)_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$  we can find a compact  $L \subset \mathbb{R}$  such that

$$\forall \varepsilon, \{g_\varepsilon \circ f_\varepsilon^{-1}(y) : y \in [-b, b]\} \cup [-b, b] \subset L,$$

and moreover it is sufficient to prove that

$$\begin{aligned} & \left( \sup_{y \in [-b, b], \eta \in L} \left| (f_\varepsilon^{-1})'(y) \frac{\partial F}{\partial \xi}(f_\varepsilon^{-1}(y), \eta, v_\varepsilon(f_\varepsilon^{-1}(y), \eta)) \right. \right. \\ & \quad \left. \left. + (f_\varepsilon^{-1})'(y) \frac{\partial v_\varepsilon}{\partial x}(f_\varepsilon^{-1}(y), \eta) \frac{\partial F}{\partial z}(f_\varepsilon^{-1}(y), \eta, v_\varepsilon(f_\varepsilon^{-1}(y), \eta)) \right| \right)_\varepsilon \in |A|. \end{aligned}$$

But it is easy to see that

$$\left( \sup_{y \in [-b, b], \eta \in L} \left| (f_\varepsilon^{-1})'(y) \frac{\partial F}{\partial \xi}(f_\varepsilon^{-1}(y), \eta, v_\varepsilon(f_\varepsilon^{-1}(y), \eta)) \right| \right)_\varepsilon \in |A|.$$

For the other term the only part needing some new explanations is to prove that

$$\left( \sup_{y \in [-b, b], \eta \in L} \left| \frac{\partial v_\varepsilon}{\partial x}(f_\varepsilon^{-1}(y), \eta) \right| \right)_\varepsilon \in |A|.$$

But here we use the fact that  $(g_\varepsilon^{-1} - f_\varepsilon^{-1})_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$  to find  $\varepsilon_0$  such that

$$\forall 0 < \varepsilon < \varepsilon_0, \|f_\varepsilon^{-1} - g_\varepsilon^{-1}\|_{[-b, b]} < 1. \quad (12)$$

We proved in the proof of theorem (20) that

$$(P_{K_\varepsilon, \alpha}(v_\varepsilon))_\varepsilon \in |A|$$

and because of (12) we have that  $f_\varepsilon^{-1}(L) \times [-b, b] \subset K_\varepsilon$ , which settles this case. For higher derivatives the reasoning involves the same estimate and presents no new obstacles. So this proves that

$$\forall \alpha, \left( P_{K_\varepsilon, \alpha} \left( \int_{f_\varepsilon^{-1}(y)}^{g_\varepsilon^{-1}(y)} \int_y^{g_\varepsilon(\xi)} F(\xi, \eta, v_\varepsilon(\xi, \eta)) d\eta d\xi \right) \right)_\varepsilon \in I_A.$$

Similar arguments apply to prove that

$$\forall \alpha, \left( P_{K_\varepsilon, \alpha} \left( \int_{f_\varepsilon^{-1}(y)}^x \int_{f_\varepsilon(\xi)}^{g_\varepsilon(\xi)} F(\xi, \eta, v_\varepsilon(\xi, \eta)) d\eta d\xi \right) \right)_\varepsilon \in I_A.$$

So we have proved that

$$\begin{aligned} & P_{K_\varepsilon, \alpha} (u_{1, \varepsilon}(x, y) - v_{1, \varepsilon}(x, y))_\varepsilon \\ & \equiv \left( \int_{f_\varepsilon^{-1}(y)}^x \int_y^{f_\varepsilon(\xi)} [F(\xi, \eta, u_\varepsilon(\xi, \eta)) - F(\xi, \eta, v_\varepsilon(\xi, \eta))] d\eta d\xi \right)_\varepsilon \mod I_A. \end{aligned}$$

We define

$$\sigma_\varepsilon(x, y) = u_\varepsilon(x, y) - v_\varepsilon(x, y) - \int_{f_\varepsilon^{-1}(y)}^x \int_y^{f_\varepsilon(\xi)} [F(\xi, \eta, u_\varepsilon(\xi, \eta)) - F(\xi, \eta, v_\varepsilon(\xi, \eta))] d\eta d\xi.$$

So by the above arguments we just proved that  $P_{K_\varepsilon, \alpha}(\sigma_\varepsilon)_\varepsilon \in I_A$ . We now define  $w_\varepsilon(x, y) = u_\varepsilon(x, y) - v_\varepsilon(x, y)$ . Keeping the same notations as in the proof of theorem (20), we want to prove that

$$\forall n, (P_{K_\varepsilon, n}(w_\varepsilon))_\varepsilon \in I_A.$$

Let us first prove that  $P_{K_\varepsilon, 0}(w_\varepsilon) \in I_A$ . First we have

$$F(\xi, \eta, u_\varepsilon(\xi, \eta)) - F(\xi, \eta, v_\varepsilon(\xi, \eta)) = w_\varepsilon(\xi, \eta) \int_0^1 \frac{\partial F}{\partial z}(\xi, \eta, u_\varepsilon(\xi, \eta) + \theta(w_\varepsilon(\xi, \eta))) d\theta,$$

then

$$w_\varepsilon(x, y) = \sigma_\varepsilon(x, y) + \int_{f_\varepsilon^{-1}(y)}^x \int_y^{f_\varepsilon(\xi)} w_\varepsilon(\xi, \eta) \left( \int_0^1 \frac{\partial F}{\partial z}(\xi, \eta, u_\varepsilon(\xi, \eta) + \theta(w_\varepsilon(\xi, \eta))) d\theta \right) d\eta d\xi.$$

Now we have

$$\forall \varepsilon, \cup_{(x, y) \in K_\varepsilon} \{(\xi, \eta) \mid \xi \in [x, f_\varepsilon^{-1}(y)], y \leq \eta \leq f_\varepsilon(\xi)\} \subset L_\varepsilon = [\alpha_{K, \varepsilon}, \beta_{K, \varepsilon}] \times [-b, b],$$

so that setting  $l_\varepsilon = \sup_{L_\varepsilon \times \mathbb{R}} \left| \frac{\partial F}{\partial z} \right|$  we have

$$|w_\varepsilon(x, y)| \leq l_\varepsilon \int_{\alpha_{K, \varepsilon}}^{\beta_{K, \varepsilon}} \int_y^{f_\varepsilon(x)} |w_\varepsilon(\xi, \eta)| d\eta d\xi + |\sigma_\varepsilon(x, y)|,$$

so

$$\forall (x, y) \in K_\varepsilon, |w_\varepsilon(x, y)| \leq l_\varepsilon \int_{\alpha_{K, \varepsilon}}^{\beta_{K, \varepsilon}} \int_y^{f_\varepsilon(x)} |w_\varepsilon(\xi, \eta)| d\eta d\xi + \|\sigma_\varepsilon\|_{K_\varepsilon}.$$

Letting  $e_\varepsilon(y) = \sup_{\xi \in [\alpha_{K, \varepsilon}, \beta_{K, \varepsilon}]} |w_\varepsilon(\xi, y)|$  we obtain

$$e_\varepsilon(y) \leq a_{K, \varepsilon} l_\varepsilon \int_y^b e_\varepsilon(\eta) d\eta + \|\sigma_\varepsilon\|_{K_\varepsilon}.$$

So applying Gronwall's lemma we finally obtain

$$e_\varepsilon(y) \leq \|\sigma_\varepsilon\|_{K_\varepsilon} \exp(a_{K,\varepsilon} l_\varepsilon(b-y)).$$

Then

$$\forall(x, y) \in K_\varepsilon, |w_\varepsilon(x, y)| \leq \|\sigma_\varepsilon\|_{K_\varepsilon} \exp(a_{K,\varepsilon} l_\varepsilon 2b)..$$

Consequently  $P_{K_\varepsilon,0}(w_\varepsilon) \in I_A$ . Which implies the 0th order estimate. According to Proposition 8, we deduce  $(w_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ ; consequently  $u$  does not depend on the choice of the representative  $(f_\varepsilon)_\varepsilon$  of the class  $f = [f_\varepsilon] \in \mathcal{G}_\tau(\mathbb{R})$ .

Now that the generalized solution is well defined let us prove that is is indeed a generalization of the non-characteristic smooth case.

From now on we will denote  $\mathcal{X}^s$  and  $\mathcal{N}^s$  the usual special algebra and its ideal (that is the scale is polynomial in  $\varepsilon$ ) and  $\mathcal{G}^s = \mathcal{X}^s/\mathcal{N}^s$  the usual special Colombeau algebra.

Let us now see why theorems 20 and 21 are a generalization of the classical case. Assume that the initial data are given along a path of equation  $y = g(x)$  where  $g$  is  $C^\infty$  and for all  $x \in \mathbb{R}$ ,  $g'(x) \neq 0$ , and that  $\varphi, \psi$  are  $C^\infty$ . We will then take  $g_\varepsilon = g$  for all  $\varepsilon$  to represent  $g$  in  $\mathcal{G}^s$ .

Let  $v$  be the classical solution then the generalized solution corresponding to  $g_\varepsilon$ , is just  $[v_\varepsilon]$  where  $v_\varepsilon = v$  for all  $\varepsilon$ . Now what we need to prove is that for any other representative  $(f_\varepsilon)_\varepsilon$  of  $g$  in  $\mathcal{A}^s$  we get a solution  $(u_\varepsilon)_\varepsilon$  which is moderate and equivalent to  $(v_\varepsilon)_\varepsilon$ . Note that this is not a consequence of theorems 20 and 21 as  $g$  is not assumed to be tempered. But actually we will see that the same proof works because  $g^{-1} \in C^\infty$  implies that  $f_\varepsilon^{-1}$  is c-bounded. We will just outline here the arguments.

First because  $f_\varepsilon^{-1}$  is c-bounded all bounds in (1) are finite and then all sets there are also compact. Now looking at the first arguments we do not need  $\varphi, \psi$  to be in  $\mathcal{O}_M(\mathbb{R})$  anymore.

Moreover  $\mathcal{C} = A/I_A$  is now overgenerated by  $(\varepsilon)_\varepsilon$ , that is  $\mathcal{A} = \mathcal{G}^s$ . Now stepping through the proof of Theorem 20 we see that because of the c-boundedness of  $f_\varepsilon^{-1}$  the moderateness of  $u_{0,\varepsilon}$  is obvious; recalling that  $D_\varepsilon$  is now replaced by a fixed compact set, the following arguments go through easily, proving the 0th estimation for  $(u_\varepsilon)_\varepsilon$ .

For the induction the c-boundedness of  $f_\varepsilon^{-1}$  (and of course of  $(f_\varepsilon)_\varepsilon$ ) ensures that all integral are done over bounded sets independent of  $\varepsilon$  which removes the need for temperateness to ensure that these integrals are moderate.

For the independance proof the lemma 23 is not needed here as both  $f_\varepsilon$  and  $f_\varepsilon^{-1}$  are c-bounded which implies that composition works as expected without the functions being tempered. The same remarks apply to the proof of Theorem 21 showing that it remains true in this context.

To summarize we have proved the

**Theorem 24.** *The solution to the Cauchy problem  $(P_C)$  in the classical case when the initial data are smooth and given along the curve  $y = f(x)$  with  $f \in C^\infty(\mathbb{R})$  and  $f'(x) \neq 0$  for all  $x \in \mathbb{R}$ , coincides with any generalized solution associated to any  $(f_\varepsilon)_\varepsilon \in \mathcal{A}^s(\mathbb{R})$  such that  $[f] = [f_\varepsilon] \in \mathcal{A}^s(\mathbb{R})$ .*

## 4 Examples

We now compute 2 examples where our method provides us with a generalized solution in characteristic situations.

**Example 25.** We consider the characteristic Cauchy problem where the initial values are smooth functions given on the characteristic curve  $\gamma$  whose equation is  $y = x^3$ . We approach  $\gamma$  by  $\gamma_\varepsilon$  whose equation is  $y = x^3 + \varepsilon x$ . We suppose that  $F = 0$ . According to the previous notations, we have to solve

$$P_{1,\varepsilon} \begin{cases} \frac{\partial^2 u_\varepsilon}{\partial x \partial y}(x, y) = 0 & (1) \\ u_\varepsilon(x, x^3 + \varepsilon x) = \varphi(x) & (2) \\ \frac{\partial u_\varepsilon}{\partial y}(x, x^3 + \varepsilon x) = \psi(x) & (3) \end{cases}$$

We can solve  $P_{1,\varepsilon}$  by putting  $u_\varepsilon(x, y) = h_\varepsilon(x) + k_\varepsilon(y)$  and we have

$$(2) \iff h_\varepsilon(x) + k_\varepsilon(x^3 + \varepsilon x) = \varphi(x), \quad (3) \iff k'_\varepsilon(x^3 + \varepsilon x) = \psi(x) \quad (V2)$$

that is to say  $k'_\varepsilon(t) = \psi(f_\varepsilon^{-1}(t))$  where  $x^3 + \varepsilon x = f_\varepsilon(x) = t$  is equivalent to  $x = f_\varepsilon^{-1}(t)$ . As  $\varphi(x) = h_\varepsilon(x) + k_\varepsilon(x^3 + \varepsilon x)$ , we deduce  $\varphi'(x) = h'_\varepsilon(x) + (3x^2 + \varepsilon)k'_\varepsilon(x^3 + \varepsilon x)$  then

$$\begin{aligned} \int_x^{f_\varepsilon^{-1}(y)} \varphi'(\xi) d\xi &= \int_x^{f_\varepsilon^{-1}(y)} h'_\varepsilon(\xi) d\xi + \int_x^{f_\varepsilon^{-1}(y)} (3\xi^2 + \varepsilon) k'_\varepsilon(\xi^3 + \varepsilon \xi) d\xi \\ &= \int_x^{f_\varepsilon^{-1}(y)} h'_\varepsilon(\xi) d\xi + \int_x^{f_\varepsilon^{-1}(y)} (3\xi^2 + \varepsilon) \psi(\xi) d\xi \\ &= [h_\varepsilon(\xi)]_x^{f_\varepsilon^{-1}(y)} + \int_0^{f_\varepsilon^{-1}(y)} (3\xi^2 + \varepsilon) \psi(\xi) d\xi - \int_0^x (3\xi^2 + \varepsilon) \psi(\xi) d\xi. \end{aligned}$$

Then

$$\varphi(f_\varepsilon^{-1}(y)) - \varphi(x) = h_\varepsilon(f_\varepsilon^{-1}(y)) - h_\varepsilon(x) + \int_0^{f_\varepsilon^{-1}(y)} (3\xi^2 + \varepsilon) \psi(\xi) d\xi - \int_0^x (3\xi^2 + \varepsilon) \psi(\xi) d\xi.$$

We deduce that

$$\varphi(f_\varepsilon^{-1}(y)) - h_\varepsilon(f_\varepsilon^{-1}(y)) + h_\varepsilon(x) = \varphi(x) + \int_0^{f_\varepsilon^{-1}(y)} (3\xi^2 + \varepsilon) \psi(\xi) d\xi - \int_0^x (3\xi^2 + \varepsilon) \psi(\xi) d\xi.$$

According to (V2), we obtain  $\varphi(f_\varepsilon^{-1}(y)) - h_\varepsilon(f_\varepsilon^{-1}(y)) = k_\varepsilon(f_\varepsilon(f_\varepsilon^{-1}(y))) = k_\varepsilon(y)$ , so,

$$k_\varepsilon(y) + h_\varepsilon(x) = \varphi(x) + \int_0^{f_\varepsilon^{-1}(y)} (3\xi^2 + \varepsilon) \psi(\xi) d\xi - \int_0^x (3\xi^2 + \varepsilon) \psi(\xi) d\xi,$$

that is to say,

$$u_\varepsilon(x, y) = \varphi(x) + \int_0^{f_\varepsilon^{-1}(y)} (3\xi^2 + \varepsilon) \psi(\xi) d\xi - \int_0^x (3\xi^2 + \varepsilon) \psi(\xi) d\xi.$$

Solve  $P_{1,\varepsilon}$ . Set

$$H(x) = \varphi(x) - \int_0^x 3\xi^2 \psi(\xi) d\xi; \quad L(x) = \int_0^x \psi(\xi) d\xi.$$

We can write

$$h_\varepsilon(x) = \varphi(x) - \int_0^x 3\xi^2 \psi(\xi) d\xi + \varepsilon \int_0^x \psi(\xi) d\xi = H(x) + \varepsilon L(x)$$

and

$$k_\varepsilon(y) = \int_0^{f_\varepsilon^{-1}(y)} (3\xi^2 + \varepsilon) \psi(\xi) d\xi.$$

So

$$u_\varepsilon = h_\varepsilon \otimes 1_y + 1_x \otimes k_\varepsilon$$

then

$$h_\varepsilon \otimes 1_y = H \otimes 1_y + \varepsilon L \otimes 1_y.$$

The class of  $(h_\varepsilon \otimes 1_y)_\varepsilon$  lies in a Colombeau algebra but it is not the case for  $(1_x \otimes k_\varepsilon)_\varepsilon$  and we have to involve a convenient  $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ -algebra.

**Example 26.** Let us take  $f(x) = \operatorname{sgn}(x)$ , and

$$f_\varepsilon(x) = \tanh(x/\varepsilon) + \alpha_\varepsilon(x) \quad \text{and} \quad g_\varepsilon(x) = \frac{2}{\pi} \arctan(x/\varepsilon) + \alpha_\varepsilon(x),$$

where  $(\alpha_\varepsilon)_\varepsilon$  is any family of smooth functions with  $\alpha_\varepsilon(\mathbb{R}) = \mathbb{R}$  and  $\alpha'_\varepsilon > 0$  such that  $(\alpha_\varepsilon)_\varepsilon \in \mathcal{N}$ . One can check that  $f_\varepsilon \rightarrow f$  and  $g_\varepsilon \rightarrow f$  (simple limits, and in fact uniform convergence over compacts not containing 0). Moreover  $f_\varepsilon, g_\varepsilon \in \mathcal{X}_\tau$  but  $f_\varepsilon - g_\varepsilon \notin \mathcal{N}$ . We compute  $u_\varepsilon = u_{0,\varepsilon}$  as  $F = 0$ , using both regularizations: first for  $f_\varepsilon$  we have  $u_\varepsilon(x, y) = \chi_\varepsilon(y) - \chi_\varepsilon(f_\varepsilon(x)) + \varphi(x)$ , so

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial x} &= -f'_\varepsilon(x) \psi(x) + \varphi'(x) \\ &\equiv -\frac{1}{\varepsilon \cosh^2(x/\varepsilon)} \psi(x) + \varphi'(x) \mod \mathcal{N}, \end{aligned}$$

then

$$\left( \frac{\partial u_\varepsilon}{\partial x} \right)_\varepsilon \sim -2\psi(0)\delta + \varphi'(x) \quad \text{as} \quad \int_{\mathbb{R}} \tanh(x) dx = 2.$$

We also have

$$\frac{\partial u_\varepsilon}{\partial y} = \psi(f_\varepsilon^{-1}(y)) \equiv \psi(\varepsilon \tanh^{-1} y) \mod \mathcal{N},$$

so

$$\left(\frac{\partial u_\varepsilon}{\partial y}\right)_\varepsilon \xrightarrow{\mathcal{C}^\infty} \psi(0)$$

Then we can conclude that

$$(u_\varepsilon)_\varepsilon \sim -2\psi(0)Y_x + \varphi(x) + \psi(0)y$$

Similar computations give that  $(v_\varepsilon)_\varepsilon$  is associated to the same distribution. As a side note this example shows that nonequivalent (but associated) deformations of the characteristic curve give generalized solutions associated to the same distribution. We do not know yet if this is a more general phenomenon.

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